

Asymptotic analysis of a fluid model modulated by an $M/M/1$ queue

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Abstract

We analyze asymptotically a differential-difference equation, that arises in a Markov-modulated fluid model. We use singular perturbation methods to analyze the problem with appropriate scalings of the two state variables. In particular, the ray method and asymptotic matching are used.

Keywords: Fluid models, $M/M/1$ queue, differential-difference equations, ray method, asymptotics.

1 Introduction

Fluid models have received much recent attention in the literature. They have been used to model statistical multiplexers in ATM (asynchronous transfer mode) networks [11], [12], [15], packet speech multiplexers [16], buffer storage in manufacturing models [19], buffer memory in store-and-forward systems [5] and high-speed digital communication networks

[9]. In these models the queue length is considered a continuous (or “fluid”) process, rather than a discrete random process that measures the number of customers. These models tend to be somewhat easier to analyze, as they allow for less randomness than more traditional queueing models.

The following is a description of a fairly general fluid model, of which many variants and special cases have been considered. Let $X(t)$ denote the amount of fluid at time t in the buffer. Furthermore, let $Z(t)$ be a continuous-time Markov process. The content of the buffer $X(t)$ is regulated (or driven) by $Z(t)$ in such a way that the *net input rate* into the buffer (i.e., the rate of change of its content) is $\eta[Z(t)]$. The function $\eta(\cdot)$ is called the *drift function*. When the buffer capacity is infinite, the dynamics of $X(t)$ are given by

$$\frac{dX}{dt} = \begin{cases} \eta[Z(t)], & X(t) > 0 \\ \max\{\eta[Z(t)], 0\}, & X(t) = 0 \end{cases} \quad (1)$$

The condition at $X(t) = 0$ ensures that the process $X(t)$ does not become negative. When the buffer capacity is finite, say B , the dynamics are given by

$$\frac{dX}{dt} = \begin{cases} \eta[Z(t)], & 0 < X(t) < B \\ \max\{\eta[Z(t)], 0\}, & X(t) = 0 \\ \min\{\eta[Z(t)], 0\}, & X(t) = B \end{cases}.$$

The condition at $X(t) = B$ prevents the buffer content from exceeding B .

In many applications, the process $Z(t)$ evolves as a finite or infinite state *birth-death process*. The description of the motion of $Z(t)$ is as follows: the process sojourns in a given state k for a random length of time, whose distribution is exponential with parameter $\lambda_k + \mu_k$. When leaving state k , the process enters either state $k + 1$ or state $k - 1$ with probabilities

$$\begin{aligned} k \rightarrow k + 1 & \quad w.p. \quad \frac{\lambda_k}{\lambda_k + \mu_k}, \quad k \in \mathcal{N} \\ k \rightarrow k - 1 & \quad w.p. \quad \frac{\mu_k}{\lambda_k + \mu_k}, \quad k \in \mathcal{N}. \end{aligned}$$

The motion is analogous to that of a random walk, except that transitions occur at random rather than fixed times. The parameters λ_k and μ_k are called, respectively, the *birth* and *death* rates. We shall assume that the birth and death rates are positive with the exception of the death rate μ_0 in the lowest state and (in case of a finite space $\mathcal{N} = \{0, 1, \dots, N\}$) the birth rate λ_N in the highest state, which are equal to zero. Also, it will be convenient to interpret λ_k and μ_k as zero if $k \notin \mathcal{N}$.

If the buffer has emptied at time t , it remains empty as long as the drift is negative. We let $\eta[Z(t)] = r_k$, given that $Z(t)$ is in state k . We shall assume throughout that $r_k \neq 0$ for all states. We shall also assume that $r_k > 0$ for at least one $k \in \mathcal{N}$, since otherwise, in the steady state, the buffer is always empty.

We let

$$\pi_k = \prod_{j=0}^{k-1} \frac{\lambda_j}{\mu_{j+1}}, \quad k \in \mathcal{N}$$

where an empty product should be interpreted as unity. The stationary probabilities p_k of the birth-death process can then be represented as

$$p_k = \frac{\pi_k}{\sum_{j \in \mathcal{N}} \pi_j}, \quad k \in \mathcal{N}.$$

When the capacity of the buffer is infinitely large, in order that a stationary distribution for $X(t)$ exists, the mean drift $\sum_{k \in \mathcal{N}} p_k r_k$ should be negative or, equivalently, the following *stability condition* should be satisfied

$$\sum_{k \in \mathcal{N}} \pi_k r_k < 0. \quad (2)$$

We let

$$\mathcal{N}^+ = \{k \in \mathcal{N} \mid r_k > 0\}, \quad \mathcal{N}^- = \{k \in \mathcal{N} \mid r_k < 0\}, \\ N_+ = |\mathcal{N}^+|, \quad N_- = |\mathcal{N}^-|$$

and since we assume that the drift in each state is nonzero, we have $\mathcal{N}^+ \cup \mathcal{N}^- = \mathcal{N}$.

Setting

$$P_k(t, x) = \Pr[X(t) \leq x, Z(t) = k]; \quad t, x \geq 0, \quad k \in \mathcal{N},$$

the Kolmogorov forward equations for the Markov process $[X(t), Z(t)]$ are given by

$$\frac{\partial P_k}{\partial t} + r_k \frac{\partial P_k}{\partial x} = \lambda_{k-1} P_{k-1} + \mu_{k+1} P_{k+1} - (\lambda_k + \mu_k) P_k, \quad k \in \mathcal{N}.$$

For the stationary distribution

$$F_k(x) \equiv \lim_{t \rightarrow \infty} P_k(t, x)$$

we have

$$r_k F'_k = \lambda_{k-1} F_{k-1} + \mu_{k+1} F_{k+1} - (\lambda_k + \mu_k) F_k, \quad k \in \mathcal{N}. \quad (3)$$

Since the buffer content is increasing whenever the drift is positive, the solution to (3) must satisfy the boundary conditions

$$F_k(0) = 0, \quad k \in \mathcal{N}^+. \quad (4)$$

This means that there is no probability mass at $x = 0$ if the drift takes you away from the boundary. Also, we must have

$$F_k(\infty) = p_k, \quad k \in \mathcal{N}, \quad (5)$$

as this is the marginal distribution of the regulating process $Z(t)$. In the finite capacity case we have the additional boundary condition

$$F_k(B) = p_k, \quad k \in \mathcal{N}^-. \quad (6)$$

This means there is no probability mass at $x = B$ if the drift moves the process from this boundary. The values of $F_k(0)$ for $k \in \mathcal{N}^-$, and of $F_k(B)$ for $k \in \mathcal{N}^+$, are not a priori known. The “half” boundary conditions (4) and (6) make these problems difficult.

The purpose of this paper is to continue our asymptotic analysis of fluid models using the ray method [6], which we successfully applied in [4] to the model first studied by Anick, Mitra and Sondhi in [2].

The paper is organized as follows. In Section 2 we state the basic equations. In Sections 3-7 we analyze these in various ranges of the state space (12). In Section 8 we study the marginal distribution. Finally, in Section 9 we summarize and interpret the results.

2 Problem statement

Let $\mathcal{N} = \{0, 1, 2, \dots\}$, and the parameters λ_k and μ_k be constant,

$$\lambda_k = \lambda, \quad \mu_k = \begin{cases} \mu, & 1 \leq k \\ 0, & k = 0 \end{cases}, \quad \rho = \frac{\lambda}{\mu} < 1.$$

The drift is taken as $r_k = k - c$, where c represents the output rate of the buffer. We assume c to be a positive non-integer number. This model corresponds to a fluid model modulated by the standard $M/M/1$ queue.

The forward Kolmogorov equations for $F_k(x)$ are then

$$(k - c)F'_k(x) = \lambda F_{k-1}(x) + \mu F_{k+1}(x) - (\lambda + \mu) F_k(x), \quad 0 \leq k \quad (7)$$

$$\mu F_0(x) = \lambda F_{-1}(x). \quad (8)$$

with boundary conditions

$$F_k(0) = 0, \quad [c] + 1 \leq k. \quad (9)$$

and limiting distribution

$$F_k(\infty) = (1 - \rho) \rho^k, \quad 0 \leq k. \quad (10)$$

Here (8) defines $F_{-1}(x)$ and this condition is equivalent to

$$-cF'_0(x) = \mu F_1(x) - \lambda F_0(x).$$

Since the buffer capacity is infinite, we need the stability condition

$$\frac{\rho}{1 - \rho} < c, \quad \text{or} \quad \rho < 1 - \frac{1}{c + 1}. \quad (11)$$

A related model, with $r_0 = \rho_0 < 0$ and $r_k = \rho > 0$ was studied in [18], [3] and [1] where a spectral representation of the solution was obtained. The same model was analyzed in [10] using continued fractions. The general case, with arbitrary r_k, μ_k and λ_k , was studied in [7] and [17] using a family of orthogonal polynomials. The fluid queue driven by a general

Markovian process was analyzed in [13]. A numerical method was presented in [8]. The full transient solution was considered in [14].

We shall analyze this model directly by using the differential-difference equation (7) satisfied by $F_k(x)$. After appropriate scalings of k and x , we analyze this equation asymptotically for $c \rightarrow \infty$, using singular perturbation methods. We also carefully treat various boundary and corner regions of the state space

$$\{(x, k) : x \geq 0, \quad 0 \leq k\}, \quad (12)$$

and indeed we show that their analysis is needed in order to obtain the asymptotic expansions away from the boundaries.

3 The ray expansion

To analyze the problem (7)-(10) for large c we introduce the scaled variables y and z , with

$$x = c^2 y, \quad k = cz, \quad z, y = O(1).$$

We define the function $G(y, z)$ and the small parameter ε by

$$\varepsilon = c^{-1}, \quad F_k(x) = G(x\varepsilon^2, k\varepsilon) = G(y, z)$$

and note that $F_{k\pm 1}(x) = G(y, z \pm \varepsilon)$.

Then (7) becomes the following equation for $G(y, z)$

$$\varepsilon(z-1)\frac{\partial G}{\partial y}(y, z) = \lambda G(y, z-\varepsilon) + \mu G(y, z+\varepsilon) - (\lambda + \mu)G(y, z) \quad (13)$$

and (9) implies that

$$G(0, z) = 0, \quad 1 < z. \quad (14)$$

Also, from (10), we have

$$F_k(\infty) = G(\infty, z) = (1 - \rho) \exp \left[\frac{1}{\varepsilon} z \ln(\rho) \right], \quad 0 < z. \quad (15)$$

To find $G(y, z)$ for ε small, we shall use the ray method. Thus, we consider solutions which have the asymptotic form

$$G(y, z) \sim \varepsilon^\nu \exp \left[\frac{1}{\varepsilon} \Psi(y, z) \right] \mathbb{K}(y, z), \quad (16)$$

where ν is a constant that must be determined. Using (16) in (13), with

$$\frac{1}{\varepsilon} \Psi(y, z \pm \varepsilon) = \frac{1}{\varepsilon} \Psi \pm \Psi_z + \frac{1}{2} \Psi_{zz} \varepsilon + O(\varepsilon^2),$$

dividing by $\exp \left[\frac{1}{\varepsilon} \Psi(y, z) \right]$, and expanding in powers of ε we obtain the *eikonal equation* for $\Psi(y, z)$

$$\mu(1 - e^q) + \lambda(1 - e^{-q}) + (z - 1)p = 0, \quad (17)$$

and the *transport equation* for $\mathbb{K}(y, z)$

$$(\mu e^q - \lambda e^{-q}) \frac{\partial \mathbb{K}}{\partial z} + (1 - z) \frac{\partial \mathbb{K}}{\partial y} + \frac{1}{2} \frac{\partial q}{\partial z} (\mu e^q + \lambda e^{-q}) \mathbb{K} = 0, \quad (18)$$

where

$$p = \frac{\partial \Psi}{\partial y}, \quad q = \frac{\partial \Psi}{\partial z}.$$

To solve (17) and (18) we use the method of characteristics, which we briefly review below.

Given the first order partial differential equation

$$\mathfrak{F}(y, z, \Psi, p, q) = 0,$$

where $p = \Psi_y$, $q = \Psi_z$, we search for a solution $\Psi(y, z)$. The technique is to solve the system of “characteristic equations” given by

$$\begin{aligned} \dot{y} &= \frac{\partial y}{\partial t} = \mathfrak{F}_p, & \dot{z} &= \mathfrak{F}_q \\ \dot{p} &= -\mathfrak{F}_y - p\mathfrak{F}_\Psi, & \dot{q} &= -\mathfrak{F}_z - q\mathfrak{F}_\Psi \\ \dot{\psi} &= p\mathfrak{F}_p + q\mathfrak{F}_q \end{aligned}$$

where we now consider $\{y, z, \psi, p, q\}$ to all be functions of the variables s and t , with $\psi(s, t) = \Psi(y, z)$. Here t measures how far we are along a particular characteristic curve or ray and s indexes them.

For the eikonal equation (17), the characteristic equations are

$$\dot{y} = z - 1 \quad (19a)$$

$$\dot{z} = \lambda e^{-q} - \mu e^q \quad (19b)$$

$$\dot{p} = 0 \quad (19c)$$

$$\dot{q} = -p \quad (19d)$$

$$\dot{\psi} = p(z - 1) + q(\lambda e^{-q} - \mu e^q). \quad (19e)$$

The particular solution is determined by the initial conditions at $t = 0$. We shall show that for this problem two different types of solutions are needed; these correspond to two distinct families of rays.

Setting $\Psi_y|_{t=0} = s$, $\Psi_z|_{t=0} = B$ and solving (19c)-(19d), yields

$$p = s, \quad q = B - st \quad (20)$$

so that Ψ_y is constant along a ray.

3.1 The rays from $(0, 1)$

We now consider the family of rays emanating from the point $y = 0, z = 1$. Evaluating (17) at $t = 0$ we get

$$\mu(1 - e^B) + \lambda(1 - e^{-B}) = 0$$

so that

$$B = 0 \quad \text{or} \quad B = \ln(\rho). \quad (21)$$

From (19b) and (20), with the initial condition $z(s, 0) = 1$ and using (21), we obtain

$$z = \frac{1}{s} [\lambda e^{-B} (e^{st} - 1) + \mu e^B (e^{-st} - 1)] + 1. \quad (22)$$

From (19a), we have

$$\dot{y}(s, 0) = z(s, 0) - 1 = 0$$

and

$$\ddot{y}(s, 0) = \dot{z}(s, 0) = \lambda e^{-B} - \mu e^B.$$

From (21) we have

$$\ddot{y}(s, 0) = \begin{cases} \lambda - \mu < 0, & B = 0 \\ \mu - \lambda > 0, & B = \ln(\rho) \end{cases}.$$

Using the initial condition $y(s, 0) = 0$ and expanding in powers of t , we get

$$y(s, t) \sim \ddot{y}(s, 0) \frac{t^2}{2}, \quad t \rightarrow 0$$

and in order to have $y > 0$ for $t > 0$ (i.e., for the rays to enter the domain $[0, \infty) \times [0, \infty)$) we need to choose

$$B = \ln(\rho) \quad (23)$$

with $B < 0$ since $\rho < 1$.

Integrating (19a) and using (22) and (23), we conclude that

$$y(s, t) = \frac{1}{s^2} [\mu(e^{st} - st - 1) + \lambda(1 - st - e^{-st})] \quad (24)$$

$$z(s, t) = \frac{1}{s} [\mu(e^{st} - 1) + \lambda(e^{-st} - 1)] + 1. \quad (25)$$

This yields the rays that emanate from $(0, 1)$ in parametric form. Several rays are sketched in Figure 1.

For $t \geq 0$ and each value of s , (24) and (25) determine a ray in the (y, z) plane, which starts from $(0, 1)$ at $t = 0$. We discuss a particular ray which can be obtained in an explicit form. For $s = 0$ we can eliminate t from (25) and obtain

$$y = Y_0(z) := \frac{(z - 1)^2}{2(\mu - \lambda)}, \quad s = 0, \quad 1 \leq z, \quad (26)$$

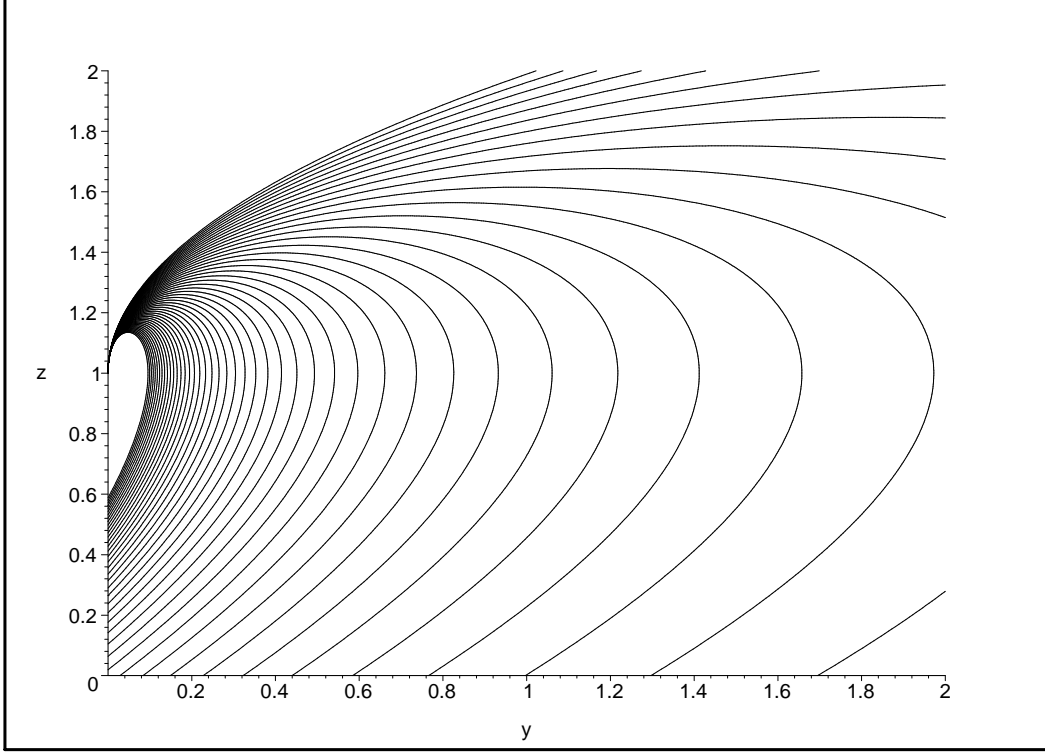


Figure 1: A sketch of the rays from $(0, 1)$.

and along this ray, t and z are related by

$$t(Y_0, z) = T_0(z) = \frac{z-1}{\mu-\lambda}, \quad s=0, \quad 1 \leq z. \quad (27)$$

For $s > 0$, we have both $y(s, t)$ and $z(s, t)$ increasing for $t > 0$. For $s < 0$ the rays reach a maximum value in z at $t = T_1$, where

$$T_1(s) = \frac{1}{2s} \ln(\rho), \quad s < 0$$

and we have

$$y(s, T_1) = \frac{1}{s^2} \left[\lambda - \mu - \frac{1}{2} (\lambda + \mu) \ln(\rho) \right] \quad (28)$$

$$z(s, T_1) = \frac{1}{s} \left[2\sqrt{\lambda\mu} - (\lambda + \mu) \right] + 1. \quad (29)$$

From (19a) we see that the maximum value in y is achieved at the same time that $z = 1$, and that occurs at $t = T_2$ with

$$T_2(s) = \frac{1}{s} \ln(\rho), \quad s < 0 \quad (30)$$

and

$$y(s, T_2) = \frac{1}{s^2} [2(\lambda - \mu) - (\lambda + \mu) \ln(\rho)].$$

Inverting the equations (24)-(25) we can write

$$s = S(y, z), \quad t = T(y, z)$$

and

$$\Psi(y, z) = \psi[S(y, z), T(y, z)], \quad \mathbb{K}(y, z) = K[S(y, z), T(y, z)].$$

We will use this notation in the rest of the article.

3.2 The functions Ψ and \mathbb{K}

From (19e) we have

$$\dot{\psi} = \mu e^{st} [1 + \ln(\rho) - ts] + \lambda e^{-st} [1 - \ln(\rho) + ts],$$

which we can integrate to get

$$\begin{aligned} \psi(s, t) &= \frac{\mu}{s} e^{st} [2 + \ln(\rho) - ts] - \frac{\lambda}{s} e^{-st} [2 - \ln(\rho) + ts] \\ &+ \psi(s, 0) - \frac{\mu}{s} [2 + \ln(\rho)] + \frac{\lambda}{s} [2 - \ln(\rho)]. \end{aligned} \quad (31)$$

Obviously, $\psi(s, 0) \equiv \psi_0$ is a constant, since all rays start at the same point. Setting $s = 0$ in (31) and using (27), we obtain

$$\psi(0, t) = \psi_0 + (z - 1) \ln(\rho)$$

and therefore, taking the limit as $t \rightarrow \infty$, we get

$$\Psi(\infty, z) = \psi_0 + (z - 1) \ln(\rho).$$

On the other hand, from (15) we have

$$\Psi(\infty, z) = z \ln(\rho)$$

and we conclude that

$$\psi_0 = \ln(\rho).$$

Solving for e^{st} in (24)-(25), we get

$$e^{st} = 1 + \frac{s}{2\mu} [z - 1 + ys + t(\lambda + \mu)], \quad e^{-st} = 1 + \frac{s}{2\lambda} [z - 1 - ys - t(\lambda + \mu)]. \quad (32)$$

Replacing (32) in (31), we obtain

$$\psi = 2ys + [\ln(\rho) - st](z - 1) + \ln(\rho). \quad (33)$$

We shall now solve the transport equation (18), which we rewrite as

$$(z-1) \frac{\partial \mathbb{K}}{\partial y} + (\lambda e^{-q} - \mu e^q) \frac{\partial \mathbb{K}}{\partial z} = \frac{1}{2} \frac{\partial q}{\partial z} (\mu e^q + \lambda e^{-q}) \mathbb{K}. \quad (34)$$

Using (24) and (25) in (34), we have

$$\frac{\partial K}{\partial t} = \frac{1}{2} \frac{\partial q}{\partial z} (\mu e^q + \lambda e^{-q}) K. \quad (35)$$

To solve (35), we need to compute $\frac{\partial q}{\partial z}$ as a function of s and t . Use of the chain rule gives

$$\begin{bmatrix} \frac{\partial y}{\partial t} & \frac{\partial y}{\partial s} \\ \frac{\partial z}{\partial t} & \frac{\partial z}{\partial s} \end{bmatrix} \begin{bmatrix} \frac{\partial t}{\partial y} & \frac{\partial t}{\partial z} \\ \frac{\partial s}{\partial y} & \frac{\partial s}{\partial z} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and hence,

$$\begin{bmatrix} \frac{\partial t}{\partial y} & \frac{\partial t}{\partial z} \\ \frac{\partial s}{\partial y} & \frac{\partial s}{\partial z} \end{bmatrix} = \frac{1}{\mathbf{J}} \begin{bmatrix} \frac{\partial z}{\partial s} & -\frac{\partial y}{\partial s} \\ -\frac{\partial z}{\partial t} & \frac{\partial y}{\partial t} \end{bmatrix}, \quad (36)$$

where the Jacobian $\mathbf{J}(s, t)$ is defined by

$$\mathbf{J}(s, t) = \frac{\partial y}{\partial t} \frac{\partial z}{\partial s} - \frac{\partial y}{\partial s} \frac{\partial z}{\partial t} = \frac{1}{s} \left[2 \frac{\partial z}{\partial t} y - (z-1)^2 \right]. \quad (37)$$

Using (36) we can show after some algebra that

$$\frac{\partial q}{\partial z} = -\frac{2y}{\mathbf{J}}, \quad (38)$$

while (32) gives

$$\mu e^q + \lambda e^{-q} = \mu + \lambda + (z-1)s.$$

Thus, the transport equation (35) becomes

$$\frac{1}{K} \frac{\partial K}{\partial t} = -\frac{y}{\mathbf{J}} [\mu + \lambda + (z-1)s]. \quad (39)$$

Using (24) and (25) in (37), we have

$$\frac{\partial \mathbf{J}}{\partial t} = \frac{2y}{s} \frac{\partial^2 z}{\partial t^2} = 2y [\mu + \lambda + (z-1)s]. \quad (40)$$

Combining (39) and (40), we obtain

$$\frac{1}{K} \frac{\partial K}{\partial t} = -\frac{1}{2\mathbf{J}} \frac{\partial \mathbf{J}}{\partial t},$$

whose solution is

$$K(s, t) = \frac{K_0(s)}{\sqrt{\mathbf{J}(s, t)}}, \quad (41)$$

where $K_0(s)$ is a function to be determined.

From (37), we have

$$\mathbf{J}(s, t) = (\mu^2 - \lambda^2) \frac{t^3}{3} + O(t^4), \quad t \rightarrow 0. \quad (42)$$

Since the Jacobian vanishes as $t \rightarrow 0$, the ray expansion ceases to be valid near the point $(0, 1)$, where a separate analysis is needed.

So far we have determined the exponent $\psi(s, t)$ and the leading amplitude $K(s, t)$ except for the function $K_0(s)$ in (41) and the power ν in (16). In Section 4 we will determine them by matching (16) to a corner layer solution valid in a neighborhood of the point $(0, 1)$.

3.3 The rays from infinity

Denoting the domain in the (y, z) plane by

$$\mathfrak{D} = [0, \infty) \times [0, \infty), \quad (43)$$

we must determine what part of \mathfrak{D} the rays from infinity fill. The expansion corresponding to these rays must satisfy the boundary condition (15). Thus, we have

$$p(\infty, z) = p_\infty = 0, \quad q(\infty, z) = q_\infty = \ln(\rho), \quad (44)$$

while (19a)-(19b) yield equations for the rays $y_\infty(t)$, $z_\infty(t)$,

$$\dot{y}_\infty = z_\infty - 1, \quad \dot{z}_\infty = \mu - \lambda \quad (45)$$

or, eliminating t from the system (45) and writing $y_\infty(t) = Y_\infty(z)$ we get

$$\frac{dY_\infty}{dz} = \frac{z - 1}{\mu - \lambda}. \quad (46)$$

Solving (45) subject to the initial condition $Y_\infty(z_0) = y_0$, where $y_0 \times z_0 = 0$, we get

$$Y_\infty(z) = y_0 + \frac{1}{2(\mu - \lambda)} [(z - 1)^2 - (z_0 - 1)^2]. \quad (47)$$

From (46), it follows that the minimum value in y occurs when $z = 1$. Hence, for y to be positive, we must have

$$y_0 > \frac{(z_0 - 1)^2}{2(\mu - \lambda)} = Y_0(z_0),$$

where $Y_0(z)$ was defined in (26). Therefore, the rays from infinity fill the region given by

$$R = \{0 \leq y, \quad 0 \leq z \leq 1\} \cup \{Y_0(z) \leq y, \quad 1 \leq z\}. \quad (48)$$

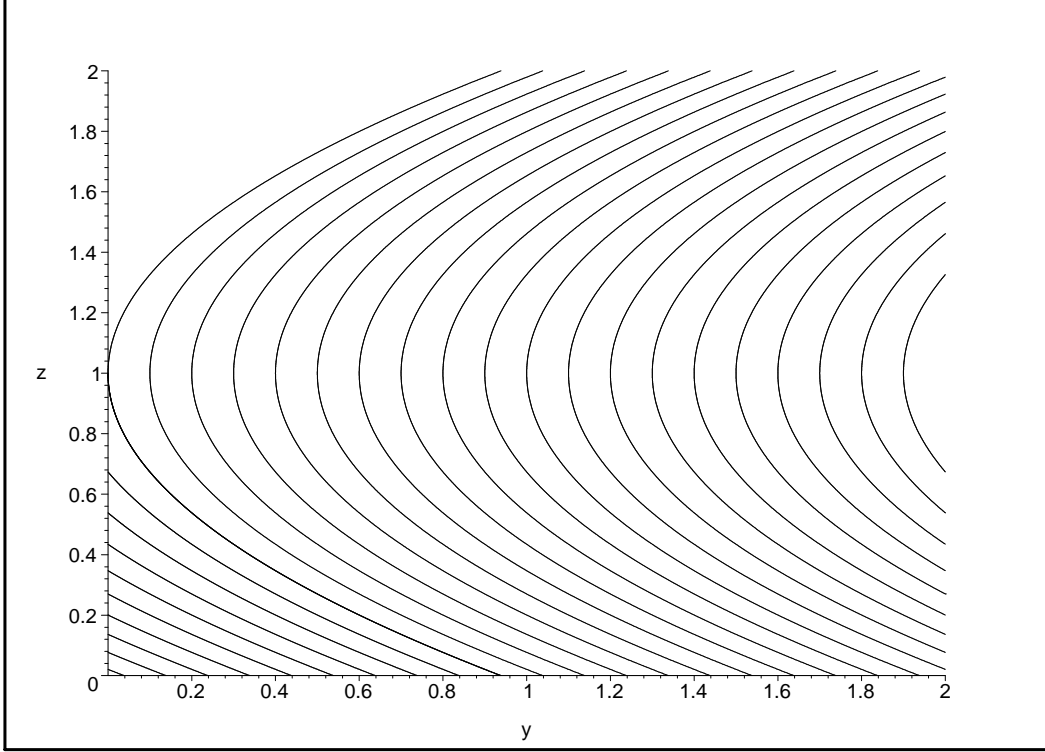


Figure 2: A sketch of the rays from infinity.

The complementary region R^C

$$R^C = \{0 \leq y < Y_0(z), \quad 1 \leq z\}, \quad (49)$$

is a *shadow* of the rays from infinity. In R^C , G is given by (16) as only the rays from $(0, 1)$ are present (see Figure 2). In the region R , both the rays coming from $(0, 1)$ and the rays coming from infinity must be taken into account. We add (16) and (15) to represent G in the asymptotic form

$$G(y, z) \sim (1 - \rho) \exp \left[\frac{1}{\varepsilon} z \ln(\rho) \right] + \varepsilon^\nu \exp \left[\frac{1}{\varepsilon} \Psi(y, z) \right] \mathbb{K}(y, z), \quad (y, z) \in R. \quad (50)$$

We can show that $z \ln(\rho) > \Psi(y, z)$ in the interior of R , so that $G(y, z) \sim G(\infty, z)$. However, in R we can write (50) as $G(y, z) - G(\infty, z) \sim \varepsilon^\nu \exp \left[\frac{1}{\varepsilon} \Psi(y, z) \right] \mathbb{K}(y, z)$.

4 The corner layer at $(0, 1)$

We determine the constant ν in (16) and the function $K_0(s)$ in (41) by considering carefully the region where the rays from $(0, 1)$ enter the domain \mathfrak{D} , and using asymptotic matching.

We introduce the stretched variable l , the function $F_l^{(1)}$ and the parameter α defined by

$$\begin{aligned} F_k(x) &= G_1(x, k - c + \alpha) = F_l^{(1)}(x) \\ l &= k - c + \alpha, \quad -\infty < l < \infty \\ \alpha &= c - \lfloor c \rfloor, \quad 0 < \alpha < 1. \end{aligned} \quad (51)$$

Note that α is the fractional part of c and l takes on integer values. Use of (51) in (7) yields the equation

$$(l - \alpha) \frac{dF_l^{(1)}}{dx} = \mu F_{l+1}^{(1)} + \lambda F_{l-1}^{(1)} - (\lambda + \mu) F_l^{(1)}, \quad x > 0, \quad l \in \mathbb{Z}. \quad (52)$$

Also, (9) gives the boundary condition

$$F_l^{(1)}(0) = 0, \quad l \geq 1 \quad (53)$$

and (15) implies that $F_k(\infty) = F_l^{(1)}(\infty)$ with

$$F_l^{(1)}(\infty) = (1 - \rho) \rho^{l-\alpha} \exp \left[\frac{1}{\varepsilon} \ln(\rho) \right]. \quad (54)$$

For a fixed l and $x \rightarrow \infty$, we approach the interior of R , where (50) applies. Thus, (54) is the asymptotic matching condition between the corner layer and the solution in R . We shall examine the matching to R^C later.

Since the problem (52)-(54) is of interest by itself, we solve it in a slightly more general setting in the next theorem.

Theorem 1 *Suppose that the function $\Phi_l(x)$ satisfies the equation*

$$(l - \alpha) \frac{d\Phi_l}{dx} = A\Phi_{l+1} + C\Phi_{l-1} - B\Phi_l, \quad x > 0, \quad l \in \mathbb{Z}, \quad (55)$$

with

$$B > 2\sqrt{AC} = \beta > 0 \quad (56)$$

and the boundary conditions

$$\Phi_l(0) = 0, \quad l \geq 1, \quad (57)$$

$$\Phi_l(\infty) = \mu_\infty r^l, \quad l \in \mathbb{Z}, \quad (58)$$

where

$$Ar^2 - Br + C = 0. \quad (59)$$

Then, Φ_l has the integral representation

$$\begin{aligned} \Phi_l(x) &= \mu_\infty \left(\frac{C}{A} \right)^{\frac{l}{2}} \sqrt{\frac{\Delta}{B}} \frac{1}{2\pi i} \int_{\text{Br}} \exp \left(x\theta + \frac{B - \Delta}{\theta} \right) \\ &\quad \times \frac{1}{\theta} \Gamma \left(1 - \alpha + \frac{B}{\theta} \right) J_{l - \alpha + \frac{B}{\theta}} \left(\frac{\beta}{\theta} \right) \left(\frac{B - \Delta}{\beta} \frac{B}{\theta} \right)^{\alpha - \frac{B}{\theta}} d\theta \end{aligned} \quad (60)$$

and the spectral representation

$$\begin{aligned} \Phi_l(x) = & \mu_\infty r^l - \mu_\infty \left(\frac{C}{A}\right)^{\frac{l}{2}} \sqrt{\frac{\Delta}{B}} \sum_{j=0}^{\infty} \frac{(j+1-\alpha)^j}{j!} \\ & \times \left(\frac{B-\Delta}{\beta}\right)^{j+1} \exp\left(x\theta_j + \frac{B-\Delta}{\theta_j}\right) J_{l-j-1}\left(\frac{\beta}{\theta_j}\right), \end{aligned} \quad (61)$$

where Br is a vertical contour in the complex θ -plane on which $\text{Re}(\theta) > 0$, $\Gamma(\cdot)$ denotes the Gamma function, $J_\nu(\cdot)$ is the Bessel function of the first kind,

$$\Delta = \sqrt{B^2 - \beta^2} = \sqrt{B^2 - 4AC}$$

and

$$\theta_j = -\frac{B}{j+1-\alpha}, \quad j = 0, 1, \dots$$

Proof. Equation (55) admits the separable solutions

$$\Phi_l(x) = e^{\theta x} h_l(\theta) \quad (62)$$

if $h_l(\theta)$ satisfies the difference equation

$$A h_{l+1} + C h_{l-1} = [(l-\alpha)\theta + B] h_l.$$

Setting

$$h_l(\theta) = \left(\frac{C}{A}\right)^{\frac{l}{2}} H_l(\theta),$$

we see that

$$H_{l+1} + H_{l-1} = \frac{2}{\beta} [(l-\alpha)\theta + B] H_l. \quad (63)$$

The only solutions to (63) which have acceptable behavior as $l \rightarrow \infty$ are of the form

$$H_l(\theta) = J_{l-\alpha+\frac{B}{\theta}}\left(\frac{\beta}{\theta}\right)$$

where $J_\nu(\cdot)$ is the Bessel function. If (62) is not to grow as $x \rightarrow \infty$, we need $\theta \leq 0$. But except when ν is an integer, the Bessel function $J_\nu(\cdot)$ is complex for negative argument. Therefore, we need

$$-\alpha + \frac{B}{\theta} = -1, -2, \dots$$

or

$$\theta_j = -\frac{B}{j+1-\alpha} < 0, \quad j \geq 0. \quad (64)$$

It follows that the general solution to (55) takes the form

$$\Phi_l(x) = \Phi_l(\infty) + \left(\frac{C}{A}\right)^{\frac{l}{2}} \sum_{j \geq 0} a_j e^{\theta_j x} J_{l-\alpha+\frac{B}{\theta_j}} \left(\frac{\beta}{\theta_j}\right)$$

or

$$\Phi_l(x) = \mu_\infty r^l + \left(\frac{C}{A}\right)^{\frac{l}{2}} \sum_{j \geq 0} a_j \exp\left(-\frac{B}{j+1-\alpha}x\right) J_{l-1-j} \left[-\frac{\beta}{B}(j+1-\alpha)\right] \quad (65)$$

where the coefficients a_j in the above (spectral) representation remain to be determined.

Taking the Laplace transform

$$\widehat{\Phi}_l(\theta) = \int_0^\infty e^{-\theta x} \Phi_l(x) dx$$

of (65) we obtain

$$\widehat{\Phi}_l(\theta) = \mu_\infty r^l \frac{1}{\theta} + \left(\frac{C}{A}\right)^{\frac{l}{2}} \sum_{j \geq 0} \frac{a_j}{\theta + \frac{B}{j+1-\alpha}} J_{j+1-l} \left[\frac{\beta}{B}(j+1-\alpha)\right]. \quad (66)$$

Thus, the only singularities of $\widehat{\Phi}_l(\theta)$ are simple poles at $\theta = 0$ and $\theta = \theta_j$, $j \geq 0$. It is well known that the Gamma function $\Gamma(z)$ has simple poles at $z = 0, -1, -2, \dots$. Hence, we shall represent $\widehat{\Phi}_l(\theta)$ as

$$\widehat{\Phi}_l(\theta) = \left(\frac{C}{A}\right)^{\frac{l}{2}} \frac{1}{\theta} \Gamma\left(\frac{B}{\theta} + 1 - \alpha\right) J_{l-\alpha+\frac{B}{\theta}} \left(\frac{\beta}{\theta}\right) f(\theta) \quad (67)$$

where $f(\theta)$ is chosen such that

$$\Gamma\left(\frac{B}{\theta} + 1 - \alpha\right) J_{l-\alpha+\frac{B}{\theta}} \left(\frac{\beta}{\theta}\right) f(\theta)$$

is analytic for $\text{Re}(\theta) > -\frac{B}{1-\alpha}$. Taking the Laplace transform in (55), we get the equation

$$(l-\alpha)\theta\widehat{\Phi}_l(\theta) = A\widehat{\Phi}_{l+1}(\theta) + C\widehat{\Phi}_{l-1}(\theta) - B\widehat{\Phi}_l(\theta), \quad l \geq 1,$$

which is satisfied by (67). By the inversion formula for the Laplace transform, we have

$$\Phi_l(x) = \left(\frac{C}{A}\right)^{\frac{l}{2}} \frac{1}{2\pi i} \int_{\text{Br}} e^{x\theta} \frac{1}{\theta} \Gamma\left(\frac{B}{\theta} + 1 - \alpha\right) J_{l-\alpha+\frac{B}{\theta}} \left(\frac{\beta}{\theta}\right) f(\theta) d\theta,$$

where Br is a vertical contour in the complex θ -plane on which $\text{Re}(\theta) > 0$.

Since the residue of $\widehat{\Phi}_l(\theta)$ at $\theta = 0$ corresponds to $\Phi_l(\infty)$, we must have

$$\left(\frac{C}{A}\right)^{\frac{l}{2}} \Gamma\left(\frac{B}{\theta} + 1 - \alpha\right) J_{l-\alpha+\frac{B}{\theta}}\left(\frac{\beta}{\theta}\right) f(\theta) \rightarrow \mu_\infty r^l$$

as $\theta \rightarrow 0$. Using the asymptotic formulas ($z \rightarrow \infty$)

$$\Gamma(a + bz) \sim \sqrt{2\pi} e^{-bz} (bz)^{a+bz-\frac{1}{2}}, \quad b > 0 \quad (68)$$

and

$$J_{a+bz}(cz) \sim \frac{1}{\sqrt{2\pi z} (b^2 - c^2)^{\frac{1}{4}}} \exp\left(z\sqrt{b^2 - c^2}\right) \left(\frac{b - \sqrt{b^2 - c^2}}{c}\right)^{a+bz}, \quad b > c > 0, \quad (69)$$

in (67), we see that

$$\Gamma\left(\frac{B}{\theta} + 1 - \alpha\right) J_{l-\alpha+\frac{B}{\theta}}\left(\frac{\beta}{\theta}\right) \sim e^{\frac{\Delta-B}{\theta}} \left(\frac{B-\Delta}{\beta}\right)^l \left(\frac{B-\Delta}{\beta} \frac{B}{\theta}\right)^{\frac{B}{\theta}-\alpha} \sqrt{\frac{B}{\Delta}}, \quad \theta \rightarrow 0$$

or, using (59),

$$\Gamma\left(\frac{B}{\theta} + 1 - \alpha\right) J_{l-\alpha+\frac{B}{\theta}}\left(\frac{\beta}{\theta}\right) \sim e^{\frac{\Delta-B}{\theta}} r^l \left(\frac{A}{C}\right)^{\frac{l}{2}} \left(\frac{B-\Delta}{\beta} \frac{B}{\theta}\right)^{\frac{B}{\theta}-\alpha} \sqrt{\frac{B}{\Delta}}, \quad \theta \rightarrow 0.$$

Therefore, we write

$$f(\theta) = \mu_\infty \sqrt{\frac{\Delta}{B}} \exp[\Upsilon(\theta)] \tilde{f}(\theta), \quad (70)$$

where

$$\Upsilon(\theta) = \frac{B-\Delta}{\theta} - \left(\frac{B}{\theta} - \alpha\right) \ln\left(\frac{B-\Delta}{\beta} \frac{B}{\theta}\right), \quad (71)$$

and $\tilde{f}(\theta)$ is entire, with $\tilde{f}(0) = 1$.

By combining the preceding results, we have

$$\Phi_l(x) = \mu_\infty \sqrt{\frac{\Delta}{B}} \left(\frac{C}{A}\right)^{\frac{l}{2}} \frac{1}{2\pi i} \int_{\text{Br}} e^{x\theta} \frac{1}{\theta} \Gamma\left(\frac{B}{\theta} + 1 - \alpha\right) J_{l-\alpha+\frac{B}{\theta}}\left(\frac{\beta}{\theta}\right) \exp[\Upsilon(\theta)] \tilde{f}(\theta) d\theta. \quad (72)$$

The boundary condition (57) implies that

$$\lim_{\theta \rightarrow \infty} [\theta \widehat{\Phi}_l(\theta)] = 0, \quad l \geq 1$$

and using the asymptotic formula

$$J_\nu(z) \sim \left(\frac{z}{2}\right)^\nu \frac{1}{\Gamma(\nu+1)}, \quad z \rightarrow 0, \quad \nu \neq -1, -2, \dots$$

in (72), we have

$$\begin{aligned} & \frac{1}{\theta} \Gamma \left(\frac{B}{\theta} + 1 - \alpha \right) J_{l-\alpha+\frac{B}{\theta}} \left(\frac{2\beta}{\theta} \right) \exp [\Upsilon(\theta)] \\ & \sim \left(\frac{1}{\theta} \right)^{l+1} \frac{\Gamma(1-\alpha)}{\Gamma(l-\alpha+1)} \left(\frac{\beta}{2} \right)^l \left(\frac{rB}{C} \right)^\alpha, \quad \theta \rightarrow \infty. \end{aligned}$$

Setting $l = 1$, we get

$$\tilde{f}(\theta) = o(\theta), \quad \theta \rightarrow \infty$$

and Liouville's theorem forces $\tilde{f}(\theta)$ to be a constant. Since $\tilde{f}(0) = 1$, we have $\tilde{f}(\theta) \equiv 1$. Thus, (72) becomes

$$\begin{aligned} \Phi_l(x) = \mu_\infty & \sqrt{\frac{\Delta}{B}} \left(\frac{C}{A} \right)^{\frac{l}{2}} \frac{1}{2\pi i} \int_{\text{Br}} \left[e^{x\theta} \frac{1}{\theta} \right. \\ & \left. \times \Gamma \left(\frac{B}{\theta} + 1 - \alpha \right) J_{l-\alpha+\frac{B}{\theta}} \left(\frac{\beta}{\theta} \right) \exp [\Upsilon(\theta)] \right] d\theta \end{aligned} \quad (73)$$

and we obtain (60).

The coefficients a_j in the spectral expansion (65) are determined from (73) by applying the residue theorem. Noting that

$$\text{Res} \left[\frac{1}{\theta} \Gamma \left(\frac{B}{\theta} + 1 - \alpha \right), \theta = \theta_j \right] = \frac{(-1)^j}{(j+1-\alpha)j!},$$

we obtain

$$a_j = \mu_\infty \sqrt{\frac{\Delta}{B}} \left(\frac{C}{A} \right)^{\frac{l}{2}} e^{x\theta_j} \frac{(-1)^j}{(j+1-\alpha)j!} J_{l-\alpha+\frac{B}{\theta_j}} \left(\frac{\beta}{\theta_j} \right) \exp [\Upsilon(\theta_j)], \quad j \geq 0. \quad (74)$$

Using (64) in (71), we get

$$\exp [\Upsilon(\theta_j)] = \exp \left(\frac{B-\Delta}{\theta_j} \right) \left(\frac{B-\Delta}{\beta} \right)^{j+1} (\alpha-j-1)^{j+1}$$

and (61) follows. ■

For the problem (52)-(54), we have

$$A = \mu, \quad B = \lambda + \mu, \quad C = \lambda, \quad r = \rho, \quad \mu_\infty = (1-\rho)\rho^{c-\alpha}, \quad \Delta = \mu - \lambda \quad (75)$$

and therefore

$$\begin{aligned} F_l^{(1)}(x) = (1-\rho) & \sqrt{\frac{\mu-\lambda}{\mu+\lambda}} \rho^{c-\alpha+\frac{l}{2}} \frac{1}{2\pi i} \int_{\text{Br}} \left[e^{x\theta} \frac{1}{\theta} \right. \\ & \left. \times \Gamma \left(\frac{\lambda+\mu}{\theta} + 1 - \alpha \right) J_{l-\alpha+\frac{\lambda+\mu}{\theta}} \left(\frac{2\sqrt{\mu\lambda}}{\theta} \right) \exp [\Lambda(\theta)] \right] d\theta, \end{aligned} \quad (76)$$

with

$$\Lambda(\theta) = \frac{2\lambda}{\theta} - \left(\frac{\lambda + \mu}{\theta} - \alpha \right) \ln \left(\sqrt{\rho} \frac{\lambda + \mu}{\theta} \right).$$

Also,

$$\begin{aligned} F_l^{(1)}(x) = (1 - \rho) \rho^{c - \alpha + \frac{l}{2}} & \left[\rho^{\frac{l}{2}} - \sqrt{\frac{\mu - \lambda}{\mu + \lambda}} \sum_{j=0}^{\infty} \frac{(j + 1 - \alpha)^j}{j!} \right. \\ & \left. \times \rho^{\frac{j+1}{2}} \exp \left(x \vartheta_j + \frac{2\lambda}{\vartheta_j} \right) J_{l-j-1} \left(\frac{2\sqrt{\mu\lambda}}{\vartheta_j} \right) \right], \end{aligned} \quad (77)$$

where

$$\vartheta_j = -\frac{\lambda + \mu}{j + 1 - \alpha}, \quad j \geq 0.$$

This completes the determination of the spectral and integral representations of $F_l^{(1)}(x)$ and hence the leading term for $F_k(x)$ in the corner region.

4.1 Matching the corner and R^C regions

In this section we shall determine the function $K_0(s)$ in (41) and the power ν in (16). We begin with the following result.

Theorem 2 *With the same hypothesis and notation as in Theorem 1, let Ω be defined by*

$$\Omega = \frac{2\Delta x}{(l - \alpha)^2} = O(1). \quad (78)$$

Then,

$$\Phi_l(x) \sim \mu_{\infty} r^l \sqrt{\frac{2B}{3\pi\Delta(l - \alpha)}} (1 - \Omega)^{-1}, \quad x \rightarrow \infty, \quad l \rightarrow \infty, \quad \Omega \rightarrow 1.$$

Proof. We set

$$\theta = \varepsilon\Theta, \quad \eta = (l - \alpha)\theta + B = (z - 1)\Theta + B, \quad \eta, \Theta = O(1), \quad \eta, \Theta > 0$$

and using (69), we obtain

$$\begin{aligned} J_{\frac{\eta}{\varepsilon\Theta}} \left(\frac{\beta}{\varepsilon\Theta} \right) & \sim \frac{\sqrt{\varepsilon\Theta}}{\sqrt{2\pi} (\eta^2 - \beta^2)^{\frac{1}{4}}} \exp \left(\frac{1}{\varepsilon\Theta} \sqrt{\eta^2 - \beta^2} \right) \left(\frac{\eta - \sqrt{\eta^2 - \beta^2}}{\beta} \right)^{\frac{\eta}{\varepsilon\Theta}} \\ & = \sqrt{\frac{\varepsilon\Theta}{2\pi p(\eta)}} \exp \left\{ \frac{1}{\varepsilon\Theta} \left[p(\eta) + \eta \ln \left(\frac{\eta - p(\eta)}{\beta} \right) \right] \right\}, \quad \varepsilon \rightarrow 0 \end{aligned}$$

with

$$p(\eta) = \sqrt{\eta^2 - \beta^2}, \quad p(B) = \Delta. \quad (79)$$

Use of (68) gives

$$\Gamma\left(\frac{B}{\varepsilon\Theta} + 1 - \alpha\right) \sim \sqrt{2\pi} \exp\left\{\frac{B}{\varepsilon\Theta} \left[\ln\left(\frac{B}{\varepsilon\Theta}\right) - 1\right]\right\} \left(\frac{B}{\varepsilon\Theta}\right)^{\frac{1}{2}-\alpha}$$

and from (71) we have

$$\exp[\Upsilon(\varepsilon\Theta)] = \exp\left[\frac{B-\Delta}{\varepsilon\Theta} - \frac{B}{\varepsilon\Theta} \ln\left(\frac{B-\Delta}{\beta} \frac{B}{\theta}\right)\right] \left(\frac{B-\Delta}{\beta} \frac{B}{\varepsilon\Theta}\right)^\alpha.$$

Therefore,

$$J_{\frac{\eta}{\varepsilon\Theta}}\left(\frac{\beta}{\varepsilon\Theta}\right) \Gamma\left(\frac{B}{\varepsilon\Theta} + 1 - \alpha\right) \exp[\Upsilon(\varepsilon\Theta)] \sim \sqrt{\frac{B}{p(\eta)}} \left(\frac{B-\Delta}{\beta}\right)^\alpha \exp\left\{\frac{1}{\varepsilon\Theta} \left[p(\eta) + \eta \ln\left(\frac{\eta-p(\eta)}{\beta}\right) - \Delta - B \ln\left(\frac{B-\Delta}{\beta}\right)\right]\right\}. \quad (80)$$

Using (80) in (73) yields, in terms of z and Ω ,

$$\Phi_l(x) \sim \mu_\infty r^\alpha \sqrt{\Delta} \left(\frac{C}{A}\right)^{\frac{z-1}{2\varepsilon}} \frac{1}{2\pi i} \int_{\text{Br}'} \frac{1}{\eta - B} \frac{1}{\sqrt{p(\eta)}} \exp\left[\frac{1}{\varepsilon} (z-1) g(\eta)\right] d\eta, \quad (81)$$

where

$$g(\eta) = \frac{(\eta - B)\Omega}{2\Delta} + \frac{1}{\eta - B} \left[p(\eta) + \eta \ln\left(\frac{\eta-p(\eta)}{\beta}\right) - \Delta - B \ln\left(\frac{B-\Delta}{\beta}\right)\right] \quad (82)$$

and Br' is a vertical contour in the complex plane with $\text{Re}(\eta) > B$. For $\varepsilon \rightarrow 0$, with Ω fixed, we can evaluate (81) by the saddle point method to get

$$\Phi_l(x) \sim \mu_\infty r^\alpha \sqrt{\Delta} \left(\frac{C}{A}\right)^{\frac{z-1}{2\varepsilon}} \sqrt{\frac{\varepsilon}{2\pi(z-1)}} \frac{1}{\eta^* - B} \exp\left[\frac{1}{\varepsilon} (z-1) g(\eta^*)\right] \frac{1}{\sqrt{p(\eta^*)g''(\eta^*)}}, \quad (83)$$

where the saddle point $\eta^*(\Omega)$ is defined by $g'(\eta^*) = 0$. Note that $\eta^*(\Omega) > B$ for $\Omega < 1$, i.e., the saddle point $\eta^*(\Omega)$ lies to the right of the pole at $\eta = B$ and the integrand is analytic for $\text{Re}(\eta) > B$.

Taking the derivative of (82), we find that

$$g'(\eta) = \frac{\Omega}{2\Delta} + \frac{1}{(\eta - B)^2} \left[\Delta + B \ln\left(\frac{B-\Delta}{\eta-p(\eta)}\right) - p(\eta)\right] \quad (84)$$

and we observe that $g'(\eta^*) = 0$ if $\eta^* = B$ and $\Omega = 1$, which implies that $\eta^*(1) = B$. To determine η^* for $\Omega \sim 1$, we use (84) and an expansion of the form

$$\eta^*(\Omega) \sim B + a_1(\Omega - 1) + a_2(\Omega - 1)^2 + a_3(\Omega - 1)^3 + \dots \quad (85)$$

Using (85) in (84) and expanding the latter in powers of $\Omega - 1$, we find that

$$a_1 = -\frac{3\Delta^2}{2B}, \quad a_2 = -\frac{27\Delta^2}{32B^3}(\Delta^2 - 3B^2)$$

and

$$\begin{aligned} g(\eta^*) &\sim \ln\left(\frac{B - \Delta}{\beta}\right) - \frac{3\Delta}{8B}(\Omega - 1)^2, \quad g''(\eta^*) \sim \frac{B}{3\Delta^3}, \\ \frac{1}{\eta^* - B} &\sim -\frac{2B}{3\Delta^2}(\Omega - 1)^{-1}, \quad \frac{1}{\sqrt{p(\eta^*)}} \sim \frac{1}{\sqrt{\Delta}}, \end{aligned}$$

from which we conclude that

$$\Phi_l(x) \sim -\mu_\infty r^l \sqrt{\frac{2B}{3\pi\Delta(l - \alpha)}}(\Omega - 1)^{-1}. \quad (86)$$

■

Using (75) in (86), we get, for $x, l \rightarrow \infty$ with $\Omega \rightarrow 1$,

$$F_l^{(1)}(x) \sim -(1 - \rho) \exp\left[\frac{z \ln(\rho)}{\varepsilon}\right] \sqrt{\frac{2(\mu + \lambda)\varepsilon}{3\pi(\mu - \lambda)(z - 1)}}(\Omega - 1)^{-1}. \quad (87)$$

This must agree with the behavior of the ray expansion in R^C as $(y, z) \rightarrow (0, 1)$. We next evaluate K and ψ in (16) near the corner $(0, 1)$. From (25), we have

$$t \sim \frac{z - 1}{\mu - \lambda} - \frac{(z - 1)^2(\mu + \lambda)}{2(\mu - \lambda)^3}s, \quad s \rightarrow 0^+. \quad (88)$$

Using (88) in (24), we obtain

$$y \sim \frac{(z - 1)^2}{2(\mu - \lambda)} - \frac{(z - 1)^3(\mu + \lambda)}{3(\mu - \lambda)^3}s, \quad s \rightarrow 0^+,$$

or, using (78),

$$s \sim -\frac{3(\mu - \lambda)^2}{2(\mu + \lambda)} \frac{\Omega - 1}{z - 1}, \quad z \rightarrow 1. \quad (89)$$

We expand (31) for small t

$$\psi(s, t) \sim \ln(\rho) [1 + (\mu - \lambda)t], \quad t \rightarrow 0,$$

which taking (88) into account gives

$$\Psi(y, z) \sim \ln(\rho) + (z-1) \ln(\rho), \quad z \rightarrow 1$$

in agreement with the exponential part of (87).

From (41)-(42), we obtain

$$K(s, t) \sim K_0(s) \sqrt{\frac{3}{(\mu^2 - \lambda^2) t^3}}, \quad t \rightarrow 0,$$

or, using (89) and (88) in the above,

$$\mathbb{K}(y, z) \sim K_0 \left(-\frac{3}{2} \frac{(\mu - \lambda)^2}{(\mu + \lambda)} \frac{\Omega - 1}{z - 1} \right) \sqrt{\frac{3(\mu - \lambda)^3}{(\mu^2 - \lambda^2)(z - 1)^3}}, \quad z \rightarrow 1. \quad (90)$$

Matching the algebraic factors in (87) and (90) yields

$$\varepsilon^\nu K_0 \left(-\frac{3}{2} \frac{(\mu - \lambda)^2}{(\mu + \lambda)} \frac{\Omega - 1}{z - 1} \right) = -\sqrt{\varepsilon} \frac{2}{3} \frac{\mu + \lambda}{(\mu - \lambda)^2} \frac{z - 1}{\Omega - 1} \sqrt{\frac{\mu - \lambda}{2\pi}} (1 - \rho),$$

which implies that

$$K_0(s) = \sqrt{\frac{\mu - \lambda}{2\pi}} \frac{1 - \rho}{s} \quad (91)$$

and $\nu = \frac{1}{2}$. This completes the determination of the asymptotic solution corresponding to rays from the point $(0, 1)$. To summarize, we have established the following.

Result 3 *The solution of (13) is asymptotically given by*

$$G(y, z) \sim \sqrt{\varepsilon} \exp \left[\frac{1}{\varepsilon} \Psi(y, z) \right] \mathbb{K}(y, z) \quad \text{in } R^C \quad (92)$$

$$G(\infty, z) - G(y, z) \sim -\sqrt{\varepsilon} \exp \left[\frac{1}{\varepsilon} \Psi(y, z) \right] \mathbb{K}(y, z) \quad \text{in } R \quad (93)$$

with

$$G(\infty, z) = (1 - \rho) \exp \left[\frac{1}{\varepsilon} z \ln(\rho) \right],$$

$$R = \{0 \leq y, \quad 0 \leq z \leq 1\} \cup \{Y_0(z) \leq y, \quad 1 \leq z\}, \quad Y_0(z) = \frac{(z-1)^2}{2(\mu - \lambda)}, \quad 1 \leq z,$$

$$\Psi(y, z) = \psi(s, t) = 2ys + [\ln(\rho) - st](z - 1) + \ln(\rho), \quad (94)$$

$$\mathbb{K}(y, z) = K(s, t) = \sqrt{\frac{\mu - \lambda}{2\pi \mathbf{J}(s, t)}} \frac{1 - \rho}{s}, \quad (95)$$

where (y, z) is related to (s, t) by (24) and (25), and $\mathbf{J}(s, t)$ was defined in (37). We note that $s < 0$ in R so that the right side of (93) is positive. This gives the leading term for the probability

$$\Pr \left[X(\infty) > x = \frac{y}{\varepsilon}, \quad Z(\infty) = k = \frac{z}{\varepsilon} \right]$$

that the buffer exceeds $x = cy$.

In the corner range where (51) applies, the leading term is given by (76) or (77).

5 Transition layer

We shall find a transition layer solution near the curve $y = Y_0(z)$ defined by (26) which separates R and R^C . On this curve $s = 0$, and hence (95) is not valid because $\mathbb{K}(y, z)$ is infinite there.

We introduce the new function $L_k(x)$ defined by

$$F_k(x) = F_k(\infty)L_k(x).$$

Then (7) yields for $L_k(x)$ the equation

$$(k - c)L'_k = \mu L_{k-1} + \lambda L_{k+1} - (\lambda + \mu)L_k$$

and the boundary condition (10) becomes

$$L_k(\infty) = 1. \tag{96}$$

In terms of the variables $y = \varepsilon^2 x$, $z = \varepsilon k$, the function $L^{(1)}(y, z) = L_k(x)$ satisfies

$$\varepsilon(z - 1) \frac{\partial L^{(1)}}{\partial y}(y, z) = \mu L^{(1)}(y, z - \varepsilon) + \lambda L^{(1)}(y, z + \varepsilon) - (\lambda + \mu)L^{(1)}(y, z)$$

and therefore, as $\varepsilon \rightarrow 0$,

$$(z - 1) \frac{\partial L^{(1)}}{\partial y} = (\lambda - \mu) \frac{\partial L^{(1)}}{\partial z} + \frac{\lambda + \mu}{2} \frac{\partial^2 L^{(1)}}{\partial z^2} \varepsilon + O\left(\frac{\partial^3 L^{(1)}}{\partial z^3} \varepsilon^2\right).$$

Introducing the stretched variable ϱ , defined by,

$$y = Y_0(z) + \sqrt{\varepsilon} \varrho = \frac{(z - 1)^2}{2(\mu - \lambda)} + \sqrt{\varepsilon} \varrho \tag{97}$$

and the function $L^{(2)}(\varrho, z) = L^{(1)}(y, z)$, we obtain for $L^{(2)}(\varrho, z)$, to leading order, the diffusion equation

$$(\lambda - \mu) \frac{\partial L^{(2)}}{\partial z} + \frac{(\lambda + \mu)}{(\lambda - \mu)^2} (z - 1)^2 \frac{\partial^2 L^{(2)}}{\partial \varrho^2} = 0. \tag{98}$$

To solve (98), we assume that $L^{(2)}(\varrho, z)$ is a function of the similarity variable $V = \frac{\varrho}{r(z)}$, and let $\mathfrak{L}(V) = L^{(2)}(\varrho, z)$, where $r(z)$ is not yet determined. From (98) we get

$$-(\lambda - \mu) r(z) r'(z) V \mathfrak{L}' + \frac{(\lambda + \mu)}{(\lambda - \mu)^2} (z - 1)^2 \mathfrak{L}'' = 0 \quad (99)$$

and (96) gives

$$\mathfrak{L}(\infty) = 1. \quad (100)$$

We can eliminate z in (99) by choosing $r(z)$ to satisfy the equation

$$-(\lambda - \mu) r(z) r'(z) = \frac{(\lambda + \mu)}{(\lambda - \mu)^2} (z - 1)^2. \quad (101)$$

We choose $r(1) = 0$, which is necessary for matching the transition layer with the corner layer solution (76), and solve (101) to obtain

$$r(z) = \sqrt{\frac{2}{3} \frac{\mu + \lambda}{(\mu - \lambda)^3}} (z - 1)^{\frac{3}{2}}. \quad (102)$$

Now (99) and (100) become

$$\mathfrak{L}'' = -V \mathfrak{L}', \quad \mathfrak{L}(\infty) = 1$$

with the solution

$$\mathfrak{L}(V) = \frac{1}{2} \left[1 + \operatorname{erf} \left(\frac{V}{\sqrt{2}} \right) \right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^V \exp \left(-\frac{1}{2} u^2 \right) du.$$

Thus, the transition layer solution for $y - Y_0(z) = O(\sqrt{\varepsilon})$ and $1 < z$ is

$$F_k(x) \sim (1 - \rho) \rho^k \frac{1}{2} \left[1 + \operatorname{erf} \left(\frac{V}{\sqrt{2}} \right) \right], \quad (103)$$

with

$$V(y, z) = \frac{y - Y_0(z)}{\sqrt{\varepsilon}} \sqrt{\frac{3}{2} \frac{(\mu - \lambda)^3}{\mu + \lambda}} (z - 1)^{-\frac{3}{2}}.$$

We can show that if (103) is expanded as $V = \frac{y - Y_0(z)}{r(z)\sqrt{\varepsilon}} \rightarrow -\infty$, the transition layer matches to the ray expansion in R^C , as given by (92), corresponding to rays emanating from $(0, 1)$.

6 The boundary layer at $z = 0$

The ray expansion in (93) does not satisfy the boundary conditions in (8). Thus, we re-examine the problem on the scale $z = O(\varepsilon)$ ($k = O(1)$), with $y > 0$. We consider solutions of (7) which have the asymptotic form

$$F_k(x) - F_k(\infty) = F_k^{(3)}(y) - F_k(\infty) \sim \varepsilon^{\nu_3} \exp\left[\frac{1}{\varepsilon}\Psi(y, 0)\right] K_k^{(3)}(y). \quad (104)$$

Using (104) in (7) and expanding in powers of ε gives, to leading order,

$$0 = \lambda K_{k-1}^{(3)} + \mu K_{k+1}^{(3)} + [\Psi_y(y, 0) - (\lambda + \mu)] K_k^{(3)},$$

or, using (20),

$$0 = \lambda K_{k-1}^{(3)} + \mu K_{k+1}^{(3)} + [S(y, 0) - (\lambda + \mu)] K_k^{(3)} \quad (105)$$

and from (8) we get

$$\rho K_{-1}^{(3)} = K_0^{(3)}. \quad (106)$$

From (25) we have, along $z = 0$,

$$\mu \xi(y) + \lambda \xi^{-1}(y) + S(y, 0) - (\lambda + \mu) = 0, \quad (107)$$

where

$$\xi(y) = \exp[S(y, 0)T(y, 0)] \quad (108)$$

and therefore we can rewrite (105) as

$$\frac{\rho K_{k-1}^{(3)} + K_{k+1}^{(3)}}{K_k^{(3)}} = \rho \xi^{-1}(y) + \xi(y). \quad (109)$$

Since $S(y, 0) < 0$ and $T(y, 0) > 0$, we see that $0 < \xi(y) < 1$ for all y . Using (108) in (24), we have

$$\mu \xi(y) - \lambda \xi^{-1}(y) + \lambda - \mu - (\lambda + \mu) \ln[\xi(y)] = S^2(y, 0)y,$$

which combined with (107) gives

$$(1 - \xi^{-1})\rho - (1 - \xi) - (\rho + 1) \ln(\xi) = \mu [(1 - \xi^{-1})\rho + (1 - \xi)]^2 y. \quad (110)$$

Solving (109) subject to (106), we obtain

$$K_k^{(3)}(y) = \left[\frac{\xi(y) - 1}{1 - \rho \xi^{-1}(y)} \xi^k(y) + \rho^k \xi^{-k}(y) \right] \overline{K}(y), \quad (111)$$

with $\overline{K}(y)$ to be determined and hence,

$$F_k^{(3)}(y) - F_k(\infty) \sim \varepsilon^{\nu_3} \exp\left[\frac{1}{\varepsilon}\Psi(y, 0)\right] \left[\frac{\xi(y) - 1}{1 - \rho \xi^{-1}(y)} \xi^k(y) + \rho^k \xi^{-k}(y) \right] \overline{K}(y). \quad (112)$$

Setting $k = z/\varepsilon$, $F_k^{(3)}(y) - F_k(\infty) = G^{(1)}(y, z)$ in (112) and letting $\varepsilon \rightarrow 0$, we get

$$G^{(1)}(y, z) \sim \varepsilon^{\nu_3} \exp \left\{ \frac{1}{\varepsilon} \Psi(y, 0) + \frac{1}{\varepsilon} z \ln \left[\frac{\rho}{\xi(y)} \right] \right\} \overline{K}(y)$$

or, using (20),

$$G^{(1)}(y, z) \sim \varepsilon^{\nu_3} \exp \left[\frac{1}{\varepsilon} \Psi(y, 0) + \frac{1}{\varepsilon} z \frac{\partial \Psi}{\partial z}(y, 0) \right] \overline{K}(y). \quad (113)$$

From (33), (37), (93) and (108) we have

$$G(y, z) - G(\infty, z) \sim \sqrt{\varepsilon} \exp \left[\frac{1}{\varepsilon} \Psi(y, 0) + \frac{1}{\varepsilon} z \frac{\partial \Psi}{\partial z}(y, 0) \right] \sqrt{\frac{\mu - \lambda}{2\pi \mathbf{J}_0(y) S(y, 0)}} (1 - \rho), \quad (114)$$

as $z \rightarrow 0$, with

$$\Psi(y, 0) = 2yS(y, 0) + \ln[\xi(y)] < 0, \quad \mathbf{J}_0(y) = 2 \left[\mu \xi(y) - \frac{\lambda}{\xi(y)} \right] y - 1 < 0,$$

$$S(y, 0) = (\lambda + \mu) - \mu \xi(y) - \lambda \xi^{-1}(y) < 0.$$

Matching (113) and (114) we conclude that

$$\nu_3 = \frac{1}{2}, \quad \overline{K}(y) = \sqrt{\frac{\mu - \lambda}{2\pi \mathbf{J}_0(y) S(y, 0)}} (1 - \rho).$$

Therefore, for $k = O(1)$ and $y > 0$, we have

$$\begin{aligned} F_k^{(3)}(y) - F_k(\infty) &\sim \sqrt{\varepsilon} \exp \left[\frac{1}{\varepsilon} \Psi(y, 0) \right] (1 - \rho) \\ &\times \left[\frac{\xi(y) - 1}{1 - \rho \xi^{-1}(y)} \xi^k(y) + \rho^k \xi^{-k}(y) \right] \sqrt{\frac{\mu - \lambda}{2\pi \mathbf{J}_0(y) S(y, 0)}}, \end{aligned} \quad (115)$$

where $\xi(y)$ is defined implicitly by (110).

7 The boundary $x = 0$

For $x = 0$ and $k \leq \lfloor c \rfloor$, the values of $F_k(0)$ can be computed from the ray expansion, since $F_k(0) - F_k(\infty) \sim \sqrt{\varepsilon} \mathbb{K}(0, z) \exp \left[\frac{1}{\varepsilon} \Psi(0, z) \right]$ is well defined. For $x = 0$ and $k \geq \lfloor c \rfloor + 1$, we have $F_k(0) = 0$ by (9). We now examine how this boundary condition is satisfied by considering the scale $y = O(\varepsilon)$ and $z > 1$. Note that this part of the boundary is in the region R^C .

From (25) we have

$$e^{st} = \frac{(z - 1)s + \mu + \lambda + \sqrt{[(z - 1)s + \mu + \lambda]^2 - 4\lambda\mu}}{2\mu}, \quad z > 1. \quad (116)$$

Using (116) in (24) we get

$$S(y, z) = \frac{z-1}{y} + \frac{1}{z-1} \left\{ (\mu + \lambda) \ln \left[\frac{\mu}{(z-1)^2} y \right] + 2\lambda \right\} + O(y \ln^2 y), \quad (117)$$

and using (117) in (116)

$$T(y, z) = \frac{1}{z-1} \ln \left[\frac{(z-1)^2}{\mu y} \right] y + O(y^2 \ln^2 y), \quad (118)$$

for $y \rightarrow 0^+$ and $z > 1$.

Using (117) and (118) in (94) and (95), we find that

$$\begin{aligned} \Psi(y, z) \sim \tilde{\Psi}(y, z) &= \ln(\rho) + (z-1) \left\{ \ln \left[\frac{\lambda y}{(z-1)^2} \right] + 2 \right\} \\ &+ \frac{1}{z-1} \left\{ (\lambda + \mu) \ln \left[\frac{\mu y}{(z-1)^2} \right] + \lambda - \mu \right\} y, \quad y \rightarrow 0. \end{aligned} \quad (119)$$

Hence, we shall consider asymptotic solutions of the form

$$F_k(x) \sim \varepsilon^\sigma \exp \left[\frac{1}{\varepsilon} \tilde{\Psi}(\varepsilon x, \varepsilon k) \right] \tilde{K}(u, \varepsilon k), \quad (120)$$

where $u = \varepsilon x$, $u = O(1)$ and σ , $\tilde{K}(u, z)$ are to be determined. Using (120) in (13) we get, to leading order,

$$(z-1) \frac{\partial \tilde{K}}{\partial z} + u \frac{\partial \tilde{K}}{\partial u} + \tilde{K} = 0. \quad (121)$$

The most general solution to (121) is

$$\tilde{K}(u, z) = \frac{1}{u} \tilde{k}(\Xi), \quad \Xi = \frac{z-1}{u}. \quad (122)$$

Hence,

$$F_k(x) \sim \tilde{G}(u, z) = \varepsilon^\sigma \exp \left[\frac{1}{\varepsilon} \tilde{\Psi}(\varepsilon u, z) \right] \frac{1}{u} \tilde{k}(\Xi). \quad (123)$$

To find $\tilde{k}(\Xi)$ and σ we will match (123) with the corner layer solution (76).

Recalling that $l - \alpha = \frac{z-1}{\varepsilon}$ and using the asymptotic formula (69) we get, as $\varepsilon \rightarrow 0$ with θ fixed

$$J_{\frac{z-1}{\varepsilon} + \frac{\lambda+\mu}{\theta}} \left(\frac{2\sqrt{\mu\lambda}}{\theta} \right) \sim \sqrt{\frac{\varepsilon}{2\pi(z-1)}} \exp \left\{ \left(\frac{z-1}{\varepsilon} + \frac{\lambda+\mu}{\theta} \right) \ln \left[\frac{\sqrt{\mu\lambda} e \varepsilon}{\theta(z-1)} \right] - \frac{(\lambda+\mu)}{\theta} \right\}. \quad (124)$$

Using (124) and writing (76) in terms of $u = \varepsilon x$ and $z = 1 + (l - \alpha)\varepsilon$, we have

$$\begin{aligned}
F_l^{(1)}(x) &\sim (1 - \rho) \sqrt{\frac{\mu - \lambda}{\mu + \lambda}} \exp \left[\left(\frac{z + 1}{2\varepsilon} - \frac{\alpha}{2} \right) \ln(\rho) \right] \sqrt{\frac{\varepsilon}{2\pi(z - 1)}} \\
&\times \frac{1}{2\pi i} \int_{\text{Br}} \left\{ \frac{1}{\theta} \exp \left[\frac{u\theta}{\varepsilon} + \left(\frac{z - 1}{\varepsilon} + \frac{\lambda + \mu}{\theta} \right) \ln \left(\frac{\sqrt{\mu\lambda}e\varepsilon}{\theta(z - 1)} \right) \right] \right. \\
&\times \Gamma \left(\frac{\lambda + \mu}{\theta} + 1 - \alpha \right) \exp \left[\frac{\lambda - \mu}{\theta} - \left(\frac{\lambda + \mu}{\theta} - \alpha \right) \ln \left(\sqrt{\rho} \frac{\lambda + \mu}{\theta} \right) \right] \left. \right\} d\theta,
\end{aligned} \tag{125}$$

To evaluate (125) asymptotically as $\varepsilon \rightarrow 0$ we shall use the saddle point method. We find that the integrand has a saddle point at $\theta = \Xi$, so that

$$\begin{aligned}
F_l^{(1)}(x) &\sim \varepsilon (1 - \rho) \sqrt{\frac{\mu - \lambda}{\mu + \lambda}} \exp \left[\left(\frac{z + 1}{2\varepsilon} - \frac{\alpha}{2} \right) \ln(\rho) \right] \frac{1}{2\pi u} \frac{1}{\Xi} \\
&\times \Gamma \left(\frac{\lambda + \mu}{\Xi} + 1 - \alpha \right) \exp \left[\frac{u\Xi}{\varepsilon} + \left(\frac{z - 1}{\varepsilon} + \frac{\lambda + \mu}{\Xi} \right) \ln \left(\frac{\sqrt{\mu\lambda}e\varepsilon}{\Xi(z - 1)} \right) \right] \\
&\times \exp \left[\frac{\lambda - \mu}{\Xi} - \left(\frac{\lambda + \mu}{\Xi} - \alpha \right) \ln \left(\sqrt{\rho} \frac{\lambda + \mu}{\Xi} \right) \right],
\end{aligned}$$

or

$$\begin{aligned}
F_l^{(1)}(x) &\sim \varepsilon (1 - \rho) \sqrt{\frac{1 - \rho}{1 + \rho}} \frac{1}{2\pi u} \frac{1}{\Xi} \Gamma \left(\frac{\lambda + \mu}{\Xi} + 1 - \alpha \right) \\
&\times \exp \left[\frac{\ln(\rho)}{\varepsilon} + \frac{u\Xi}{\varepsilon} \ln \left(\frac{\lambda e^2 \varepsilon}{\Xi^2 u} \right) + \alpha \ln \left(\frac{\lambda + \mu}{\Xi} \right) \right] \\
&\times \exp \left\{ \frac{\lambda + \mu}{\Xi} \ln \left[\frac{\varepsilon}{\Xi u (\rho + 1)} \right] + \frac{2\lambda}{\Xi} \right\}.
\end{aligned} \tag{126}$$

Writing (123) in terms of Ξ , we obtain

$$\begin{aligned}
\tilde{G}(u, u\Xi + 1) &= \varepsilon^\sigma \exp \left[\frac{\ln(\rho)}{\varepsilon} + \frac{u\Xi}{\varepsilon} \ln \left(\frac{\lambda e^2 \varepsilon}{u\Xi^2} \right) \right] \\
&\times \exp \left[\frac{(\lambda + \mu)}{\Xi} \ln \left(\frac{\mu \varepsilon}{u\Xi^2} \right) + \frac{(\lambda - \mu)}{\Xi} \right] \frac{1}{u} \tilde{k}(\Xi).
\end{aligned} \tag{127}$$

Matching (126) with (127), we have

$$\begin{aligned}
\tilde{k}(\Xi) &= (1 - \rho) \sqrt{\frac{1 - \rho}{1 + \rho}} \frac{1}{2\pi} \frac{1}{\Xi} \Gamma \left(\frac{\lambda + \mu}{\Xi} + 1 - \alpha \right) \\
&\times \exp \left[\alpha \ln \left(\frac{\lambda + \mu}{\Xi} \right) + \frac{\lambda + \mu}{\Xi} \ln \left(\frac{e\Xi}{\lambda + \mu} \right) \right]
\end{aligned}$$

and $\sigma = 1$. Therefore, for $1 < z$,

$$\begin{aligned}\tilde{G}(u, u\Xi + 1) &= \varepsilon (1 - \rho) \sqrt{\frac{1 - \rho}{1 + \rho}} \frac{1}{2\pi} \frac{1}{\Xi} \frac{1}{u} \Gamma \left(\frac{\lambda + \mu}{\Xi} + 1 - \alpha \right) \\ &\times \exp \left[\frac{\ln(\rho)}{\varepsilon} + \frac{u\Xi}{\varepsilon} \ln \left(\frac{\lambda e^2 \varepsilon}{\Xi^2 u} \right) + \alpha \ln \left(\frac{\lambda + \mu}{\Xi} \right) \right] \\ &\times \exp \left\{ \frac{\lambda + \mu}{\Xi} \ln \left[\frac{\varepsilon}{\Xi u (\rho + 1)} \right] + \frac{2\lambda}{\Xi} \right\},\end{aligned}$$

or

$$\begin{aligned}\tilde{G}(u, z) &= \varepsilon (1 - \rho) \sqrt{\frac{1 - \rho}{1 + \rho}} \frac{1}{2\pi} \frac{1}{z - 1} \Gamma \left[\frac{(\lambda + \mu) u}{z - 1} + 1 - \alpha \right] \\ &\times \exp \left\{ \frac{\ln(\rho)}{\varepsilon} + \frac{z - 1}{\varepsilon} \ln \left[\frac{\lambda e^2 \varepsilon u}{(z - 1)^2} \right] + \alpha \ln \left[\frac{(\lambda + \mu) u}{z - 1} \right] \right\} \\ &\times \exp \left\{ \frac{(\lambda + \mu) u}{z - 1} \ln \left[\frac{\varepsilon}{(\rho + 1)(z - 1)} \right] + \frac{2\lambda u}{z - 1} \right\},\end{aligned}\tag{128}$$

Note that from (128) we have $\tilde{G}(u, \varepsilon k) = O(u^{k - \lfloor c \rfloor})$, as $u \rightarrow 0$, $k \geq \lfloor c \rfloor + 1$.

8 The marginal distribution

We will now find the equilibrium probability that the buffer content exceeds x ,

$$M(x) = \Pr[X(\infty) > x] = 1 - \sum_{k=0}^{\infty} F_k(x)\tag{129}$$

for various ranges of x .

8.1 Approximation for $x = O(1)$

In this region we shall use the spectral representation of the corner layer solution. Using the generating function

$$\sum_{i=-\infty}^{\infty} J_i(x) z^i = \exp \left[\frac{x}{2} \left(z - \frac{1}{z} \right) \right],$$

in the form

$$\exp \left[\frac{1 - \rho}{\rho + 1} (j + 1 - \alpha) \right] = (\sqrt{\rho})^{-(j+1)} \sum_{l=-\infty}^{\infty} J_{l-(j+1)} \left[-\frac{2\sqrt{\rho}}{\rho + 1} (j + 1 - \alpha) \right] (\sqrt{\rho})^l,$$

we obtain from (77)

$$M(x) \sim M^{(1)}(x) = (1 - \rho) \rho^{c-\alpha+1} \sqrt{\frac{1-\rho}{1+\rho}} \sum_{j \geq 0} \frac{(j+1-\alpha)^j}{j!} \rho^j \times \exp \left[-\frac{x(\lambda + \mu)}{j+1-\alpha} + \frac{1-3\rho}{\rho+1} (j+1-\alpha) \right]. \quad (130)$$

8.2 Approximation for $x = O(\varepsilon^{-2}) = O(c^2)$

We shall now use the asymptotic solution in the region R , as given by (93). We have

$$M(x) \sim M^{(2)}(y) = -\sum_{k=0}^{\infty} G(y, k\varepsilon) \sim -\frac{1}{\sqrt{\varepsilon}} \int_0^{\infty} \exp \left[\frac{1}{\varepsilon} \Psi(y, z) \right] \mathbb{K}(y, z) dz. \quad (131)$$

To evaluate (131) as $\varepsilon \rightarrow 0$, we use the Laplace method. From (20) we get

$$\Psi_z(y, z) = q = 0 \quad \Leftrightarrow \quad st = \ln(\rho) \quad (132)$$

and therefore the main contribution to (131) comes from $z = 1$ and we obtain

$$M^{(2)}(y) \sim -\frac{\sqrt{2\pi}}{\sqrt{-\Psi_{zz}(y, 1)}} \exp \left[\frac{1}{\varepsilon} \Psi(y, 1) \right] \mathbb{K}(y, 1).$$

Using (116) in (37)-(38) and (94)-(95), we obtain

$$\begin{aligned} \Psi_{zz}(y, 1) &= \frac{S(y, 1)}{\mu - \lambda}, \quad \mathbb{K}(y, 1) = \frac{1 - \rho}{2S(y, 1)} \sqrt{\frac{-S(y, 1)}{\pi y}}, \\ \Psi(y, 1) &= 2yS(y, 1) + \ln(\rho), \end{aligned} \quad (133)$$

while (24) gives

$$S(y, 1) = -\frac{\zeta}{\sqrt{y}}, \quad (134)$$

with

$$\zeta = \sqrt{2(\lambda - \mu) - (\lambda + \mu) \ln(\rho)}.$$

Thus,

$$M^{(2)}(y) \sim \sqrt{\frac{\mu - \lambda}{2}} \frac{1 - \rho}{\zeta} \exp \left[\frac{1}{\varepsilon} (-2\zeta \sqrt{y} + \ln \rho) \right]. \quad (135)$$

9 Summary and discussion

In most of the domain $\mathfrak{D} = \{(y, z) : y, z \geq 0\}$, the asymptotic expansion of $F_k(x) = G(y, z)$ is given by

$$G(y, z) \sim \sqrt{\varepsilon} \exp \left[\frac{1}{\varepsilon} \Psi(y, z) \right] \mathbb{K}(y, z) \quad \text{in } R^C \quad (136)$$

or

$$G(\infty, z) - G(y, z) \sim -\sqrt{\varepsilon} \exp \left[\frac{1}{\varepsilon} \Psi(y, z) \right] \mathbb{K}(y, z) \quad \text{in } R. \quad (137)$$

If we consider the continuous part of the density, given by

$$f_k(x) = F'_k(x) = \varepsilon^2 \frac{\partial G}{\partial y}(y, z), \quad x > 0,$$

the transition between R and R^C disappears, and we have

$$f_k(x) \sim \varepsilon^{\frac{3}{2}} \Psi_y(y, z) \exp \left[\frac{1}{\varepsilon} \Psi(y, z) \right] \mathbb{K}(y, z) = \varepsilon^{\frac{3}{2}} \exp \left[\frac{1}{\varepsilon} \psi(s, t) \right] sK(s, t), \quad (138)$$

everywhere in the interior of \mathfrak{D} . Note that $\mathbb{K}(y, z)$ becomes infinite along $y = Y_0(z)$ (i.e., $s = 0$), but the product $\Psi_y(y, z) \mathbb{K}(y, z)$ remains finite.

The asymptotic expansion of the boundary probabilities $F_k(0)$, $k \leq \lfloor c \rfloor$ can be obtained by setting $y = 0$ in (137). This expression can be used to estimate the difference

$$F_k(\infty) - F_k(0) = \Pr \left[X(\infty) > 0, \quad Z(\infty) = k = \frac{z}{\varepsilon} \right]$$

which is exponentially small for $\varepsilon \rightarrow 0$. Also, for a fixed $z \in [0, 1)$, $f_k(x)$ is maximal at $x = 0$ (see Figure 3). In other words, if $k < c$, the buffer will most likely be empty.

For a fixed $z > 1$, $f_k(x)$ is peaked along the curve $y = Y_0(z)$ (see Figure 4). To see this better, we can use (27), (36) and (37) in (138), obtaining

$$f_k(x) \sim \varepsilon^{\frac{3}{2}} \frac{1-\rho}{\sqrt{2\pi}} \sqrt{\chi(z)} \exp \left\{ \frac{1}{\varepsilon} \left[z \ln(\rho) - \chi(z) (y - Y_0)^2 \right] \right\}, \quad z > 1$$

with

$$\chi(z) = \frac{3(\mu - \lambda)^3}{(\mu + \lambda)(z - 1)^3},$$

or equivalently

$$f_k(x) \sim (1 - \rho) \rho^k \sqrt{\frac{\chi(\frac{k}{c})}{2\pi c^3}} \exp \left\{ -\frac{\chi(\frac{k}{c})}{c^3} \left[x - c^2 Y_0 \left(\frac{k}{c} \right) \right]^2 \right\}, \quad z > 1.$$

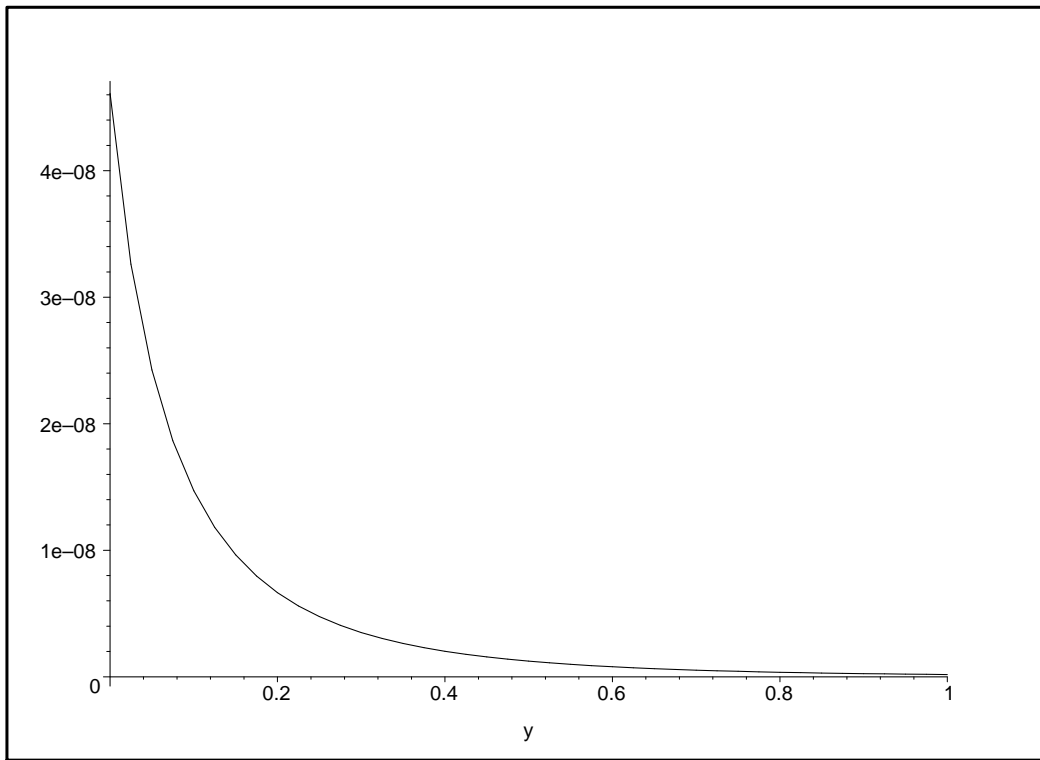


Figure 3: A plot of $\varepsilon^{\frac{3}{2}}\Psi_y(y, z) \exp\left[\frac{1}{\varepsilon}\Psi(y, z)\right] \mathbb{K}(y, z)$, with $\varepsilon = 0.1, \lambda = 0.3145, \mu = 0.8473$ and $z = 0.5$.

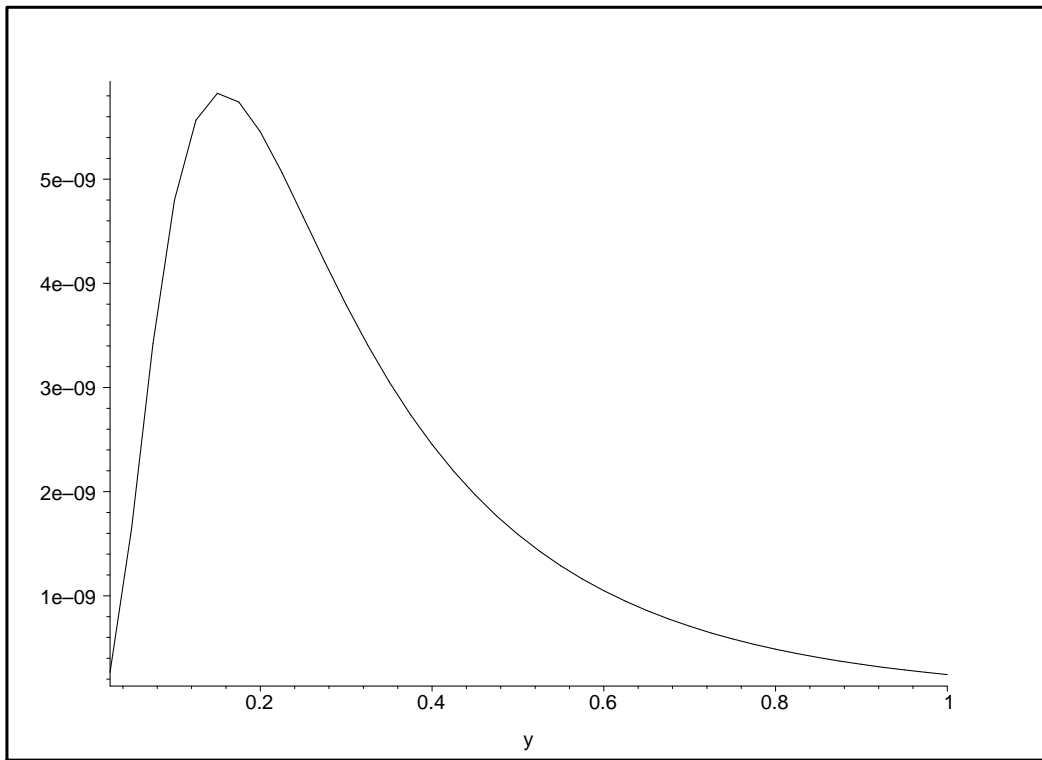


Figure 4: A plot of $\varepsilon^{\frac{3}{2}}\Psi_y(y, z) \exp\left[\frac{1}{\varepsilon}\Psi(y, z)\right] \mathbb{K}(y, z)$, with $\varepsilon = 0.1, \lambda = 0.3145, \mu = 0.8473$ and $z = 1.5$.

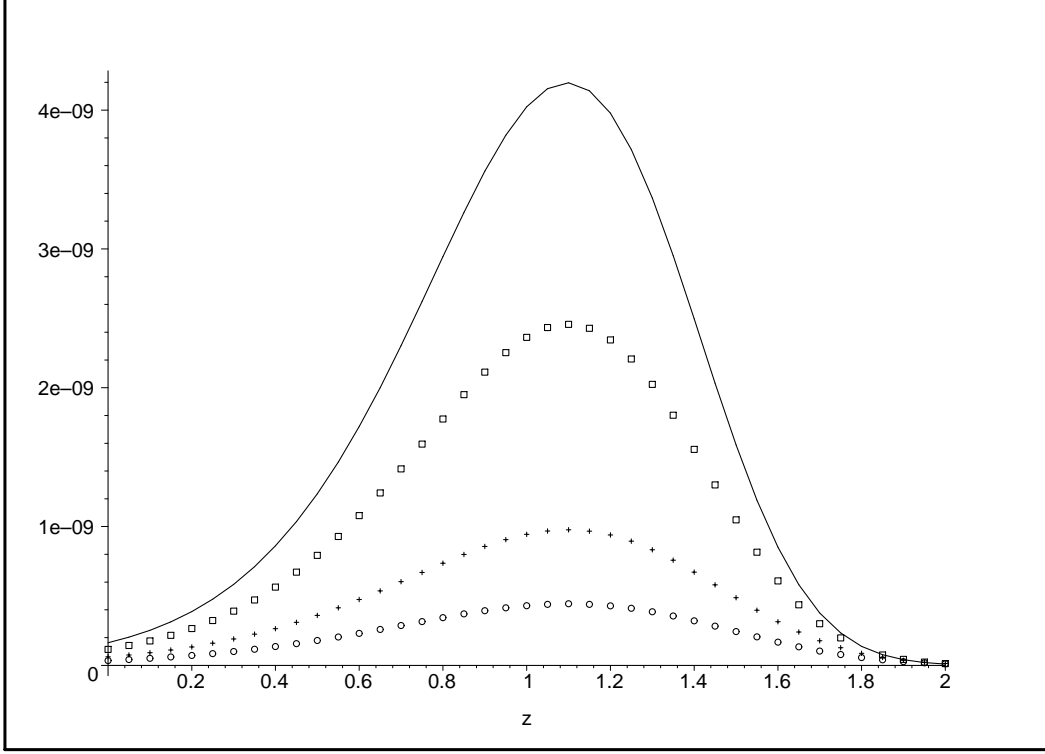


Figure 5: A plot of $\varepsilon^{\frac{3}{2}} \Psi_y(y, z) \exp\left[\frac{1}{\varepsilon} \Psi(y, z)\right] \mathbb{K}(y, z)$, with $\varepsilon = 0.1, \lambda = 0.3145, \mu = 0.8473$ for $y = 0.5$ (solid line), $y = 0.6$ (\square), $y = 0.8$ ($+++$) and $y = 1$ (ooo).

This means that given $k > c$ active sources, the most likely value of the buffer will be

$$x = c^2 Y_0 \left(\frac{k}{c} \right).$$

For a fixed $x \geq 0$, $f_k(x)$ achieves its maximum around $z = 1$ (see Figure 5). To find an expression for $f_k(x)$ when z is close to 1, we use (133) and obtain, for fixed $y > 0$,

$$f_k(x) \sim \varepsilon^{\frac{3}{2}} \frac{1 - \rho}{2} \sqrt{\frac{\zeta}{\pi}} y^{-\frac{3}{4}} \exp \left\{ \frac{1}{\varepsilon} \left[\ln \rho - 2\zeta \sqrt{y} - \frac{\zeta (z - 1)^2}{2\sqrt{y}(\mu - \lambda)} \right] \right\},$$

or

$$f_k(x) \sim \frac{1}{2} (1 - \rho) \rho^c \sqrt{\frac{\zeta}{\pi}} x^{-\frac{3}{4}} \exp \left[-2\zeta \sqrt{x} - \frac{\zeta (k - c)^2}{2\sqrt{x}(\mu - \lambda)} \right].$$

Below we summarize the various boundary, corner and transition layer corrections to the results in (93) and (92):

1. $k = l + c - \alpha$, $x = O(1)$:

$$F_l^{(1)}(x) = (1 - \rho) \sqrt{\frac{\mu - \lambda}{\mu + \lambda}} \rho^{c - \alpha + \frac{l}{2}} \\ \times \frac{1}{2\pi i} \int_{\text{Br}} e^{x\theta} \frac{1}{\theta} \Gamma\left(\frac{\lambda + \mu}{\theta} + 1 - \alpha\right) J_{l - \alpha + \frac{\lambda + \mu}{\theta}}\left(\frac{2\sqrt{\mu\lambda}}{\theta}\right) \exp[\Lambda(\theta)] d\theta,$$

where $J(\cdot)$ denotes the Bessel function, $\Gamma(\cdot)$ the Gamma function, Br is a vertical contour in the complex plane with $\text{Re}(s) > 0$ and

$$\alpha = c - \lfloor c \rfloor \in (0, 1), \quad \rho = \frac{\lambda}{\mu} < 1, \quad \Lambda(\theta) = \frac{2\lambda}{\theta} - \left(\frac{\lambda + \mu}{\theta} - \alpha\right) \ln \left[\sqrt{\rho} \frac{\lambda + \mu}{\theta} \right].$$

2. $y - Y_0(z) = O(\sqrt{\varepsilon})$, $1 < z$:

$$F_k(x) \sim (1 - \rho) \rho^k \frac{1}{2} \left[1 + \text{erf} \left(\frac{V}{\sqrt{2}} \right) \right],$$

with

$$V(y, z) = \frac{y - Y_0(z)}{\sqrt{\varepsilon}} \sqrt{\frac{3}{2} \frac{(\mu - \lambda)^3}{\mu + \lambda}} (z - 1)^{-\frac{3}{2}}, \quad Y_0(z) = \frac{(z - 1)^2}{2(\mu - \lambda)}, \quad 1 < z$$

3. $k = O(1)$

$$F_k^{(3)}(y) - F_k(\infty) \sim \sqrt{\varepsilon} \exp \left[\frac{1}{\varepsilon} \Psi(y, 0) \right] \\ \times \left[\frac{\xi(y) - 1}{1 - \rho \xi^{-1}(y)} \xi^k(y) + \rho^k \xi^{-k}(y) \right] \sqrt{\frac{\mu - \lambda}{2\pi \mathbf{J}_0(y) S(y, 0)}} (1 - \rho), \\ \Psi(y, 0) = 2yS(y, 0) + \ln[\xi(y)] < 0, \quad \mathbf{J}_0(y) = 2 \left[\mu \xi(y) - \frac{\lambda}{\xi(y)} \right] y - 1 < 0, \\ S(y, 0) = (\lambda + \mu) - \mu \xi(y) - \lambda \xi^{-1}(y) < 0, \\ (1 - \xi^{-1}) \rho - (1 - \xi) - (\rho + 1) \ln(\xi) = \mu \left[(1 - \xi^{-1}) \rho + (1 - \xi) \right]^2 y.$$

4. $y = \varepsilon u$, $u = O(1)$, $1 < z$

$$F_k(x) \sim \varepsilon (1 - \rho) \sqrt{\frac{1 - \rho}{1 + \rho}} \frac{1}{2\pi} \frac{1}{z - 1} \Gamma \left[\frac{(\lambda + \mu)u}{z - 1} + 1 - \alpha \right] \\ \times \exp \left\{ \frac{\ln(\rho)}{\varepsilon} + \frac{z - 1}{\varepsilon} \ln \left[\frac{\lambda e^2 \varepsilon u}{(z - 1)^2} \right] + \alpha \ln \left[\frac{(\lambda + \mu)u}{z - 1} \right] \right\} \\ \times \exp \left\{ \frac{(\lambda + \mu)u}{z - 1} \ln \left[\frac{\varepsilon}{(\rho + 1)(z - 1)} \right] + \frac{2\lambda u}{z - 1} \right\}.$$

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